DIMENSIONAL ANALYSIS

Recap: Preliminaries

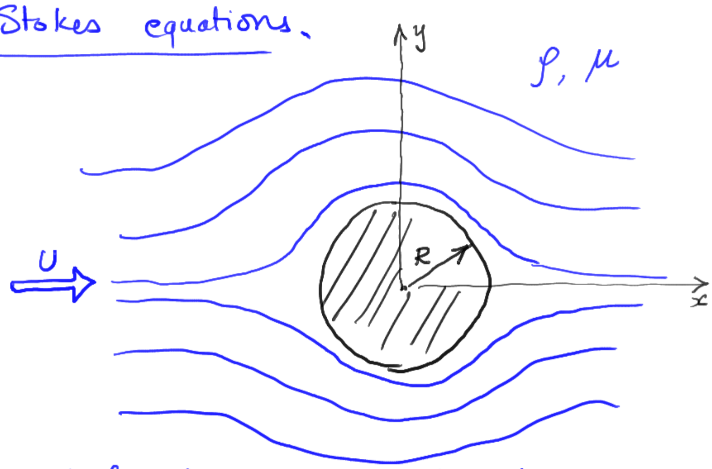
- Dimension of product is the product of dimensions
- Only quantities with equal dimensions can be added or equated.

List of topics

- Non-dimensionalizing the Navier-Stokes equations.
- Deducing dependence on parameters without solving PDEs.

1. Non-dimensionalizing the Navier-Stokes equations.

E.g. Flow past a cylinder.



Find the drag (per unit depth) on a cylinder of radius R immersed in an incompressible fluid (density ρ, viscosity μ) flowing with speed U.

Mathematical formulation

Fluid velocity $\underline{u} = (u,v)$ and pressure p satisfy the incompressible N-S equations,

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Solve these equations w/ boundary conditions

$$\underline{u}\,(|\underline{x}| \to \infty, t) = U\,\hat{e}_x$$
$$\underline{u}\,(|\underline{x}| = R, t) = \underline{0}. \qquad \text{no-slip}$$

Suppose a steady solution is found. Then the drag per unit length is

$$D = \int_{\partial\Omega} \hat{e}_x \cdot \underline{\underline{T}} \cdot \hat{n}\, dA \qquad \text{where} \quad \underline{\underline{T}} = -p\underline{\underline{I}} + \mu\left(\nabla\underline{u} + \nabla\underline{u}^T\right).$$

$$\boxed{\text{Given parameters}: \quad \rho, U, R, \mu.}$$

Use these to rescale dependent and independent variables.

$$t = (R/U)\,\tilde{t}$$
$$(u,v) = U\,(\tilde{u},\tilde{v}), \qquad p = \rho U^2\,\tilde{p} \qquad \left[\text{Also possible} \quad p = \frac{\mu U}{R}\tilde{p}\right].$$
$$(x,y) = R\,(\tilde{x},\tilde{y})$$

dimensionless variables

characteristic scales.

$$\frac{\mu U/R^2}{\rho U^2/R} = \frac{\mu}{\rho U R} \Bigg/ \rho\left[\frac{U}{R/U}\frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{U^2}{R}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{U^2}{R}\tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}}\right] = \left(\frac{\rho U^2}{R}\right)\left(-\frac{\partial \tilde{p}}{\partial \tilde{x}}\right) + \frac{\mu U}{R^2}\left(\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right).$$

Rewrite as
$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{1}{Re}\left(\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right).$$

Similarly
$$\frac{\partial \tilde{v}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{y}} + \frac{1}{Re}\left(\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}\right).$$

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0.$$

with
$$\underline{\tilde{u}}(|\underline{\tilde{x}}| \to \infty) = 1\,\hat{e}_x.$$
$$\underline{\tilde{u}}(|\underline{\tilde{x}}| = 1) = 0$$

Nearly the same as the unscaled version.

$$\underline{\underline{T}} = \rho U^2\left[-\tilde{p}\underline{\underline{I}} + \frac{\mu}{\rho U R}\left(\nabla\underline{\tilde{u}}^T + \nabla\underline{\tilde{u}}\right)\right] \quad \text{and} \quad D = \rho U^2 R \int_{\partial\Omega} \underline{\underline{\tilde{T}}}\cdot\hat{n}\, d\tilde{A}$$

$$\frac{\underline{\underline{T}}}{\rho U^2} = \underline{\underline{\tilde{T}}} = -\tilde{p}I + \frac{1}{Re}\left(\nabla\underline{\tilde{u}}^T + \nabla\underline{\tilde{u}}\right).$$

Define $Re = \frac{\rho U R}{\mu}$ = Reynolds number = $\frac{\text{inertial forces}}{\text{viscous forces}}$.
(after Osborne Reynolds).

- All flows with the same Reynolds numbers have the same rescaled flow.
- $\frac{D}{\rho U^2 R} = \int_{\partial\Omega} \underline{\underline{\tilde{T}}}\cdot\hat{n}\, d\tilde{A}$ depends only on Re.

Terminology: Drag coefficient $C_D = \frac{D}{\frac{1}{2}\rho U^2 R} = F(Re)$ alone.

Conclusion: Non-dimensionalization can reduce the number of independent parameters.

2. Deducing dependence on parameters withouts solving PDEs:

Drag on a cylinder depends on ρ, μ, R, U.

$$D = f(\rho, \mu, R, U).$$

Now use arbitrariness in the system of units.

$$\tilde{l} = \alpha l, \quad \tilde{t} = \beta t, \quad \tilde{m} = \gamma m$$

Then $\tilde{\rho} = \frac{\tilde{m}}{\tilde{l}^3} = \frac{\gamma}{\alpha^3}\frac{m}{l^3} = \frac{\gamma}{\alpha^3}\rho.$

$$\tilde{\mu} = \frac{\gamma}{\alpha\beta}\mu, \quad \tilde{U} = \frac{\alpha}{\beta}U, \quad \tilde{R} = \alpha R \quad \text{and} \quad \tilde{D} = \frac{\gamma}{\beta^2}D.$$

The drag relation must also hold in the transformed system.

$$\tilde{D} = f(\tilde{\rho}, \tilde{\mu}, \tilde{R}, \tilde{U})$$

Transforming back

$$\frac{\gamma}{\beta^2} D = f\left(\frac{\gamma}{\alpha^3}\rho, \frac{\gamma}{\alpha\beta}\mu, \alpha R, \frac{\alpha}{\beta} U\right).$$ valid for any α, β, γ.

Choose $\frac{\gamma}{\alpha^3}\rho = 1$, $\frac{\alpha}{\beta}U = 1$, and $\alpha R = 1$,

i.e., $\underbrace{\alpha = \frac{1}{R}}_{\text{length scale}}$, $\underbrace{\beta = \frac{U}{R}}_{\text{time scale}}$ and $\gamma = \frac{1}{\rho R^3}$.

Working in a system of units in which:
- fluid has unit density
- flows w/ unit speed
- cylinder has unit radius.

$$\frac{D}{\rho U^2 R} = f\left(1, \frac{1}{Re}, 1, 1\right) = \frac{1}{2} F(Re)$$ ... same as before.

We can say a little more about the behaviour of $F(Re)$ in the limits $Re \to 0$ (viscous limit) and $Re \to \infty$ (inertial limit).

$$Re = \frac{\text{inertial forces}}{\text{viscous forces}}.$$

- Inertial limit: $D = f_I(\rho, U, R)$ ... viscosity "negligible".

  $= \frac{1}{2} C_D \rho U^2 R.$ where C_D is dimensionless

  $\Rightarrow F(Re) = C_D =$ constant as $Re \to \infty$.

- Viscous limit: $D = f_v(\mu, U, R)$ ... inertia "negligible".

  $= C_v \mu U$ where C_v is dimensionless.

  $F(Re) = \frac{2C_v}{Re}$ as $Re \to 0$.