The Bernoulli Equation

Recap: Navier-Stokes equations, the incompressible version

$$\rho\left[\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right] = -\nabla p + \rho g + \mu\nabla^2\underline{u} \quad \text{... momentum}$$

$$\nabla\cdot\underline{u} = 0 \quad \text{... mass conservation}$$

Also, from hydrostatics: $p = \rho g\cdot\underline{x} + p_0$, and the Euler equation.

Objective: An explicit expression for p, if possible.

Use the identity,

$$\underline{u}\times\underline{\omega} = \nabla\left(\frac{|\underline{u}|^2}{2}\right) - \underline{u}\cdot\nabla\underline{u} \quad \text{(see Aside for derivation)}$$

where $\underline{\omega} = \nabla\times\underline{u}$ = vorticity

(see Deformation of infinitesimal fluid elements)

$$\rho\left[\frac{\partial u}{\partial t} + \nabla\left(\frac{|\underline{u}|^2}{2}\right) - \underline{u}\times\underline{\omega}\right] = -\nabla p + \rho g + \mu\nabla^2\underline{u}.$$

OR (assuming g = const.)

Aside:

$$[\underline{u}\times(\nabla\times\underline{u})]_i = \varepsilon_{ijk}\, u_j\, [\nabla\times\underline{u}]_k$$

$$= \varepsilon_{ijk}\,\varepsilon_{kmn}\, u_j \frac{\partial u_n}{\partial x_m}$$

$$= [\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}]\, u_j \frac{\partial u_n}{\partial x_m}$$

$$= u_j \frac{\partial u_j}{\partial x_i} - u_j \frac{\partial u_i}{\partial x_j}.$$

$$= \frac{1}{2}\frac{\partial (u_j u_j)}{\partial x_i} - u_j\frac{\partial u_i}{\partial x_j}$$

$$= \left[\nabla\left(\frac{|\underline{u}|^2}{2}\right) - \underline{u}\cdot\nabla\underline{u}\right]_i$$

$$\boxed{\rho\frac{\partial \underline{u}}{\partial t} + \nabla\left[\rho\frac{|\underline{u}|^2}{2} + p - \rho g\cdot\underline{x}\right] = \mu\nabla^2\underline{u} + \rho\,\underline{u}\times\underline{\omega} \quad \text{...} \star}$$

CASE I: Steady inviscid flow ... i.e. $\frac{\partial \underline{u}}{\partial t} = 0$ and $\mu = 0$.

$\star$ becomes.

$$\nabla\underbrace{\left[\rho\frac{|\underline{u}|^2}{2} + p - \rho g\cdot\underline{x}\right]}_{\tilde{B}\,=\,\text{Bernoulli function}} = +\rho\,(\underline{u}\times\underline{\omega}).$$

Interpretation:

- $\tilde{B}$ changes only along a direction perpendicular to both $\underline{u}$ & $\underline{\omega}$.
- $\tilde{B}$ constant along a streamline. (Streamline everywhere tangent to $\underline{u}$.)
- $\tilde{B}$ constant along a "vortex line". (Vortex line everywhere tangent to $\underline{\omega}$.)

$$\boxed{\tilde{B} = \frac{1}{2}\rho\frac{|\underline{u}|^2}{2} + p - \rho g\cdot\underline{x} = \text{constant along a streamline.}}$$

Steady inviscid flow.

CASE II : Potential flow, i.e. $\underline{u} = \nabla\phi$ for some ϕ = velocity potential

$$\Rightarrow \nabla \underbrace{\left[\rho \frac{\partial\phi}{\partial t} + \rho\frac{|\underline{u}|^2}{2} + p - \rho \underline{g}\cdot\underline{x}\right]}_{\hat{B} = \text{Bernoulli function}} = +\rho\left[\cancelto{0}{\underline{u}\times\underline{\omega}}\right] + \cancelto{0}{\mu\nabla^2\underline{u}}$$

$\underline{\omega} = \nabla\times\underline{u}$
$= \nabla\times\nabla\phi$
$= \underline{0}$

$\nabla\cdot\underline{u} = 0$
$\Rightarrow \nabla^2\phi = 0$
Thus,
$\nabla^2\underline{u} = \nabla^2\nabla\phi = \nabla(\nabla^2\phi) = 0.$

Interpretation:

- $\hat{B}$ is a constant everywhere in the flow (but may vary with time)

i.e. $\boxed{\hat{B} = \rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho|\underline{u}|^2 + p - \rho\underline{g}\cdot\underline{x} = f(t) \text{ alone.}}$

Next video – Applications of Bernoulli Equation.